semiconductors with dielectric constants of 10–20, has been shown to also yield reliable data for a plastic material, polyethylene, with a dielectric constant of only about 2. Since, for most polymers, the dispersion is very small in the radio-frequency to microwave range and the microwave measurement is both very simple and accurate, the interference method may find technical application in material characterization. Prerequisites are, of course, a low loss tangent and, hence, good transmission through the thick sample.

ACKNOWLEDGMENT

The chromium–gold evaporation of the samples by Dr. V. Schlosser is gratefully acknowledged. The author is indebted to Dr. W. Meyer, Max-Planck-Institut für Polymerforschung, Mainz, and Dr. Köhler, BASF, Ludwigshafen, for supplying the Lupolen 1810 A material and to Dr. Hellmann, Deutsches Kunststoff-Institut, Darmstadt, for pressing plane-parallel plates from Lupolen 1810 A. The author wishes to express his deep gratitude to Prof. Dr. G. Harbeke, Paul Scherrer Institute, Zürich, for donating the samples and for a revision of the manuscript.

REFERENCES

[1] A. C. Lynch in *High-Frequency Dielectric Measurements*, J. Chamberlain and G. W. Chantry, Eds. Guildford, Surrey, U. K.: IPC Science and Technology Press, 1973, pp. 34–37.
[2] K. Seeger, "Microwave dielectric constants of silicon, gallium arsenide, and quartz," *J. Appl. Phys.*, vol. 63, pp. 5439–5443, 1988.
[3] K. Seeger, "Temperature dependence of the dielectric constants of semi-insulating III–V compounds," *Appl. Phys. Lett.*, vol. 54, pp. 1268–1269, 1989.
[4] B. Carlowitz, *Kunststoff Tabellen*, 3rd ed. Munich: Hanser Verlag, 1986, p. 89.
[5] C. B. Duke *et al.*, "Electronic structure of pendant-group polymers: Molecular-ion states and dielectric properties of poly (2 vinyl pyridine)," *Phys. Rev. B*, vol. 18, pp. 5717–5739, 1978.

Equivalence of Propagation Characteristics for the Transmission-Line Matrix and Finite-Difference Time-Domain Methods in Two Dimensions

N. R. S. Simons and E. Bridges

Abstract —**In previous papers an equivalence between the TLM and FD-TD methods has been established by altering the definitions of field components and operation of the TLM algorithm such that the appropriate finite-difference expressions are satisfied. In this paper the equivalence of propagation characteristics for the TLM and FD-TD methods in two dimensions is discussed. Propagation analysis of a TLM shunt node complete with permittivity and loss stubs, and dispersion analysis of the two-dimensional FD-TD method in an arbitrary medium are performed and yield dispersion relations. The relations are identical when the FD-TD method is operated at the upper limit of its stability range.**

Manuscript received February 6, 1990; revised September 17, 1990.
The authors are with the Department of Electrical and Computer Engineering, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2.
IEEE Log Number 9041086.

I. Introduction

The finite-difference time-domain (FD–TD) method, devised by Yee [1], is based on central difference approximations of Maxwell's curl equations. The method has been successfully applied to a variety of problems, as summarized by Taflove and Umashankar [2]. The transmission-line matrix (TLM) method, pioneered by Johns and Beurle [3], is based on a physical model of wave propagation and has also been extensively applied to the analysis of electromagnetic field problems [4]. Both are time-domain numerical techniques capable of simulating Maxwell's equations in arbitrary media.

Previous papers by Johns and Butler [5] and Johns [6] have shown that TLM models for diffusion and electromagnetic field problems can be operated in such a way that they are equivalent to finite-difference methods. The implications of operating the TLM models under these circumstances have been discussed by Johns [6], with comments by Gwarek [7]. In [5] and [6], the TLM method is formulated in terms of global scattering and connection matrices. Johns has stated that a great deal of flexibility exists in the operation of a TLM algorithm and the definitions of field components in terms of the pulses traveling along the elemental transmission lines. Johns describes the manner in which a mesh of three-dimensional expanded nodes can be operated such that it satisfies the three-dimensional Yee algorithm [6]. A similar analysis has been performed in which a mesh of unloaded two-dimensional shunt nodes is shown to satisfy the Yee algorithm in two dimensions [8]. The equivalence is obtained by altering the definitions for field quantities in the TLM mesh.

The purpose of this paper is to show that, mathematically, the propagation characteristics of the two methods are identical under certain circumstances, regardless of the definitions of field quantities and operation of the TLM model.

II. Dispersion Relation of FD–TD Method

For finite-difference approximations of the wave equation, the dispersion relation is obtained by substituting the mathematical representation of a plane wave into the difference approximation of the wave equation. A general discussion of dispersion in finite-difference models of the wave equation has been provided by Trefethen [9]. For two-dimensional field distributions (independent of the z direction, and selecting $H_z = 0$) Maxwell's curl equations can be combined to yield the following wave equation for E_z:

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = \sigma\mu \frac{\partial E_z}{\partial t} + \epsilon\mu \frac{\partial^2 E_z}{\partial t^2} \tag{1}$$

for a medium of permittivity ϵ, permeability μ and conductivity σ. Expression (1) can be discretized using central difference approximations to obtain

$$\begin{aligned}&\frac{E_z^{t_0}(x_0+\Delta l, y_0) - 2E_z^{t_0}(x_0, y_0) + E_z^{t_0}(x_0-\Delta l, y_0)}{\Delta l^2}\\ &\quad + \frac{E_z^{t_0}(x_0, y_0+\Delta l) - 2E_z^{t_0}(x_0, y_0) + E_z^{t_0}(x_0, y_0-\Delta l)}{\Delta l^2}\\ &= \sigma\mu \frac{E_z^{t_0+\Delta t}(x_0, y_0) - E_z^{t_0-\Delta t}(x_0, y_0)}{2\Delta t}\\ &\quad + \epsilon\mu \frac{E_z^{t_0+\Delta t}(x_0, y_o) - 2E_z^{t_o}(x_o, y_o) + E_z^{t_o-\Delta t}(x_o, y_o)}{\Delta t^2}.\end{aligned} \tag{2}$$

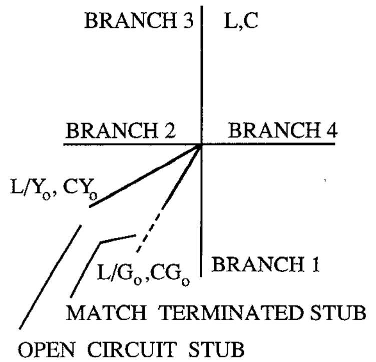

Fig. 1. Two-dimensional shunt node complete with permittivity and loss stubs.

A solution of (1) can be written as

$$E_z = e^{j\omega t + \gamma \cos\phi x + \gamma \sin\phi y} \tag{3}$$

which represents a plane wave traveling through the medium characterized by a complex propagation constant $\gamma = \alpha + j\beta$ at an angle ϕ to the x axis. Inserting solution (3) into the difference approximation (2), the following relationship is obtained:

$$\sinh^2 \frac{\gamma \cos\phi \Delta l}{2} + \sinh^2 \frac{\gamma \sin\phi \Delta l}{2} = j\frac{\sigma\mu\Delta l^2}{4\Delta t}\sin\omega\Delta t - \epsilon\mu\frac{\Delta l^2}{\Delta t^2}\sin^2\frac{\omega\Delta t}{2}. \tag{4}$$

For the case of a nonconductive medium ($\alpha = 0$), (4) reduces to the familiar result [9]

$$\sin^2\frac{\beta\cos\phi\Delta l}{2} + \sin^2\frac{\beta\sin\phi\Delta l}{2} = \epsilon\mu\frac{\Delta l^2}{\Delta t^2}\sin^2\frac{\omega\Delta t}{2}. \tag{5}$$

III. Propagation Analysis of a TLM Shunt Node

Brewitt-Taylor and Johns have applied the concept of determining the dispersion relation to TLM models and have referred to it as propagation analysis [10]. The same technique that is used to calculate dispersion relations is applied, but instead of substituting the analytical solution of the lossy wave equation into a difference approximation, it is substituted into an equation governing the behavior of voltages on a transmission line matrix. Brewitt-Taylor and Johns apply propagation analysis to compare two- and three-dimensional lumped element and transmission line models for lossless homogeneous media. In this section propagation analysis of a two-dimensional shunt node complete with permittivity and conductivity stubs is performed.

The two-dimensional node complete with permittivity and loss stubs is shown in Fig. 1. The elemental transmission lines (branches 1, 2, 3, 4) have a distributed inductance, L, and distributed capacitance, C. As introduced by Akhtarzad and Johns [11], the open circuit stub of characteristic admittance Y_0 is used to model permittivity:

$$\epsilon_r = 2\left(1 + \frac{Y_0}{4}\right) \tag{6}$$

and the match terminated stub of characteristic admittance G_0 is used to model conductivity:

$$\sigma = \frac{G_0}{\Delta l}\sqrt{\frac{\epsilon_0}{\mu_0}}. \tag{7}$$

The node can be analyzed using simple transmission line theory and superposition. Exciting the node with v_1 and suppressing nodes 2, 3, and 4, the equivalent circuit of the node is shown in Fig. 2. The load impedance, Z_L, is made up of three short circuit stubs of length Δl and unit characteristic impedance, one open circuit stub of length $\Delta l/2$ and characteristic admittance Y_0, and a match terminated line with characteristic admittance G_0, all in parallel. The impedance Z_L is expressed as

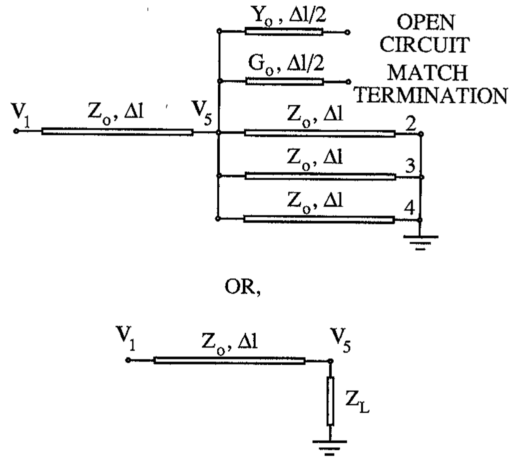

Fig. 2. Equivalent circuit of two-dimensional shunt node with branches 2, 3, and 4 shorted and branch 1 excited with a voltage v_1.

$$Z_L = \left(\frac{3}{j\tan\beta_l\Delta l} + jY_0\tan\frac{\beta_l\Delta l}{2} + G_0\right)^{-1} \tag{8}$$

where β_l denotes the phase constant for propagation along the elemental transmission lines of the model. Transmission line theory can be used to write

$$v_1 = v_5^1\left(\cos\beta_l\Delta l + \frac{j\sin\beta_l\Delta l}{Z_L}\right) \tag{9}$$

where v_5^1 is the voltage at the center of the node with all nodes but 1 suppressed. Applying the principle of superposition, the voltage v_5 can be expressed as a summation of the voltages v_5^n as

$$v_5 = \sum_{n=1}^{4} v_5^n \tag{10}$$

where v_5^n is the voltage obtained at node 5 caused by a voltage v_n applied to node n with all other nodes suppressed. Owing to the symmetry of the node, expression (10) can be written as

$$\sum_{n=1}^{4} v_n = v_5\left(\cos\beta_l\Delta l + \frac{j\sin\beta_l\Delta l}{Z_L}\right) \tag{11}$$

and substituting expression (8) into (11) yields

$$\sum_{n=1}^{4} v_n = v_5\left(4\cos\beta_l\Delta l - 2Y_0\sin^2\frac{\beta_l\Delta l}{2} + jG_0\sin\beta_l\Delta l\right). \tag{12}$$

Equation (12) governs the behavior of voltages on the transmission line model. The transmission line matrix approximates a uniform propagating medium. An ideal voltage wave traveling through space at an angle ϕ to the x axis is given by

$$V_z = V_0 e^{j\omega t + \gamma\cos\phi x + \gamma\sin\phi y} \tag{13}$$

where γ is the complex phase constant for the voltage wave, given by $\alpha + j\beta$. Substituting (13) into (12) yields the following expression:

$$\sinh^2 \frac{\gamma \cos \phi \Delta l}{2} + \sinh^2 \frac{\gamma \sin \phi \Delta l}{2} = j\frac{G_0}{4} \sin \beta_l \Delta l - 2\left(1 + \frac{Y_0}{4}\right) \sin^2 \frac{\beta_l \Delta l}{2}. \quad (14)$$

If a medium with no conductivity is considered ($\alpha = G_0 = 0$), expression (14) can be reduced to

$$\sin^2 \frac{\beta \cos \phi \Delta l}{2} + \sin^2 \frac{\beta \sin \phi \Delta l}{2} = 2\left(1 + \frac{Y_0}{4}\right) \sin^2 \frac{\beta_l \Delta l}{2}. \quad (15)$$

On the right-hand side of (14) and (15), the term $\beta_l \Delta l$ can be replaced by $\omega \Delta t$, since the phase constant β_l refers to propagation along the elemental transmission lines of the model.

IV. Equivalence of Propagation Characteristics

Although the propagation characteristics of both the two-dimensional TLM and FD–TD methods have been examined in the past, the equivalence between the two methods has not been established. Expression (4) and (14) are the dispersion relations for the FD–TD and TLM methods, respectively, in a lossy medium. Expressions (5) and (15) are the dispersion relations for the FD–TD and TLM methods, respectively, in a lossless medium. Many different choices exist in the possible forms of the conditions of equivalence. If the traditional TLM definitions for the material constants are maintained (given by (6) and (7)), the TLM and FD–TD methods correspond under the following conditions:

$$\epsilon_r = 2\left(1 + \frac{Y_0}{4}\right) \quad (16a)$$

$$\sigma = \frac{G_0}{\Delta l} \quad (16b)$$

$$\mu_r = 1 \quad (16c)$$

and

$$\frac{\Delta t}{\Delta l} = 1. \quad (16d)$$

Note that a TLM mesh represents a normalized simulation space in which the vacuum relative permittivity is 2, and the vacuum relative permeability is 1. Therefore, the maximum phase velocity of waves traveling through a two-dimensional mesh of shunt nodes is $1/\sqrt{2}$.

If the TLM mesh is considered to represent a medium having a vacuum relative permittivity and permeability of 1, then the material parameters in the normalized TLM mesh would be

$$\epsilon_r = \left(1 + \frac{Y_0}{4}\right) \quad (17a)$$

and

$$\mu_r = 1. \quad (17b)$$

Equation (15) could be rewritten as

$$\sin^2 \frac{\beta \cos \phi \Delta l}{2} + \sin^2 \frac{\beta \sin \phi \Delta l}{2} = 2\epsilon_r \mu_r \sin^2 \frac{\omega \Delta t}{2}. \quad (18)$$

The above expression and the lossless FD–TD dispersion relation are identical if

$$2\epsilon_r \mu_r = \epsilon \mu \frac{\Delta l^2}{\Delta t^2} \quad (19)$$

or

$$c_{\max} = \frac{1}{\sqrt{2}} \frac{\Delta l}{\Delta t}. \quad (20)$$

The above relation is the FD–TD stability criterion at the upper limit of stability. Therefore, the FD–TD and TLM algorithms have identical propagation characteristics if the FD–TD algorithm is operated at the upper limit of its stability range.

Operating the FD–TD algorithm at its stability limit has benefits in that the amount of dispersion introduced by the algorithm is minimum for all directions of propagation. In addition, for certain test cases, simulations indicate that accuracy is near optimal [12].

V. Conclusions

In Section II, the dispersion relation for the two-dimensional Yee algorithm was presented. This relation has been derived by many different authors in the past. Brewitt-Taylor and Johns have applied a similar concept, referred to as propagation analysis, to determine dispersion relations of transmission line models. In Section III, propagation analysis was applied to determine the dispersion relation for a two-dimensional shunt node complete with permittivity and loss stubs, which has not been previously reported. The dispersion relations are shown to be equivalent when the FD–TD algorithm is operated at the upper limit of its stability range. This equivalence has been established regardless of the definitions for field quantities and operation of a two-dimensional mesh of shunt nodes. Basic equivalences such as the one reported in this paper will help researchers working with either method, and serve as a starting point for future comparisons.

Johns has discussed the significance of comparisons and the philosophy behind the FD–TD and TLM methods [6]. The two methods are distinct in that they are based on different ideas. The TLM method is based on a discrete model for wave propagation realized as a mesh of intersecting transmission lines; the FD–TD method is based on mathematical finite differencing. It is interesting to find that these distinct concepts produce numerical methods that can have identical wave propagation characteristics.

References

[1] K. S. Yee, "Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media," *IEEE Trans. Antennas Propagat.*, vol. AP-14, pp. 302–307, May 1966.

[2] A. Taflove and K. Umashankar, "The finite-difference time-domain (FD–TD) method for electromagnetic scattering and interaction problems," *J. Electromagn. Waves and Appl.*, vol. 1, no. 3, pp. 243–267, 1987.

[3] P. B. Johns and R. L. Beurle, "Numerical solution of two dimensional scattering problems using a transmission line matrix," *Proc. Inst. Elec. Eng.*, vol. 118, no. 9, pp. 1203–1208, Nov. 1971.

[4] W. J. R. Hoefer, "The transmission-line matrix method—Theory and applications," *IEEE Trans. Microwave Theory Tech.* vol. MTT-33, pp. 882–893, Oct. 1985.

[5] P. B. Johns and G. Butler, "The consistency and accuracy of the TLM method for diffusion and its relationship to existing methods," *Int. J. Numer. Meth. Eng.*, vol. 19, pp. 1549–1554, 1983.

[6] P. B. Johns, "On the relationship between TLM and finite difference methods for Maxwell's equations," *IEEE Trans. Microwave Theory Tech.* vol. MTT-35, pp. 60–61, Jan. 1987.

[7] W. K. Gwarek, "Comments on 'The relationship between TLM and finite difference methods for Maxwell's equations,'" *IEEE Trans. Microwave Theory Tech.*, vol. MTT-35, pp. 872–873, Sept. 1987.

[8] N. R. S. Simons, "Application of the transmission-line matrix method to open region field problems," M.Sc. thesis, University of Manitoba, 1989.

[9] L. N. Trefethen, "Group velocity in finite difference schemes," *SIAM Rev.*, no. 24, pp. 113–136, 1982.

[10] C. R. Brewitt-Taylor and P. B. Johns, "On the construction and numerical solution of transmission-line and lumped network models

of Maxwell's equations," *Int. J. Numer. Meth. Eng.*, vol. 19, pp. 13–30, 1980.

[11] S. Akhtarzad and P. B. Johns, "Generalized elements for the TLM method of numerical analysis," *Proc. Inst. Elec. Eng.*, vol. 122, no. 12, pp. 1349–1352, Dec. 1975.

[12] I. S. Kim and W. J. R. Hoefer, "Effect of the stability factor on the accuracy of two-dimensional TD–FD simulation," in *Proc. 1989 IEEE AP-S Symp.*, 1989, pp. 1108–1111.

Analysis of a Transition Between Rectangular and Circular Waveguides

B. N. Das and P. V. D. Somasekhar Rao

Abstract —**This paper presents analysis of a transition between rectangular and circular waveguides coupled by a rectangular slot in a metallic wall of finite thickness in the common transverse cross section. Expressions for VSWR and admittance are obtained using a moment method formulation with entire basis and testing functions. Numerical data on the variation of input VSWR with frequency are obtained and a comparison between the theoretical and experimental results is presented. The variations in the values of minimum VSWR with change in slot dimensions are also studied.**

I. Introduction

Excitation of a circular waveguide from a rectangular waveguide has attracted the attention of scientific workers for a long time [1]–[3]. The transition which has been suggested is designed in such a way that there is a transformation of cross section from rectangular on one side to circular on the other side. The transition designed for better matching [2], [3] is quite bulky apart from the complexity in mechanical fabrication.

Investigations of coupling between waveguides through apertures in the form of a rectangular slot in the common transverse cross section have been reported [4]–[6]. To the best of the authors' knowledge, no data on the performance characteristics of this type of junction are available in the literature.

In the present work, investigations are carried out for a junction (transition) between rectangular and circular waveguides coupled through a rectangular slot. Analysis based on the method of moments with entire sinusoidal basis and testing functions taking into account the effect of the finite thickness of the transverse metallic wall in which the slot is milled is similar to that in [7]. The expressions for the elements of matrices to be inverted are different for the problem under investigation.

Expressions for the coefficient, VSWR and normalized shunt admittance seen by the rectangular waveguide are derived. A comparison between the theoretical and experimental results on the variation of input VSWR with frequency is presented for a rectangular slot of length 1.7 cm and width 0.107 cm in a metal plate of thickness 0.09144 cm. Variations in the values of minimum VSWR with change in slot parameters are also studied.

Manuscript received February 22, 1990; revised August 21, 1990.

B. N. Das is with the Department of Electronics and Electrical Communications Engineering, Indian Institute of Technology, Kharagpur 721302 India.

P. V. D. Somasekhar Rao is with the Department of Electronics and Communications Engineering, Jawaharlal Nehru Technological University, Mahaveer Marg Hyderabad, 500 028 India.

IEEE Log Number 9041074.

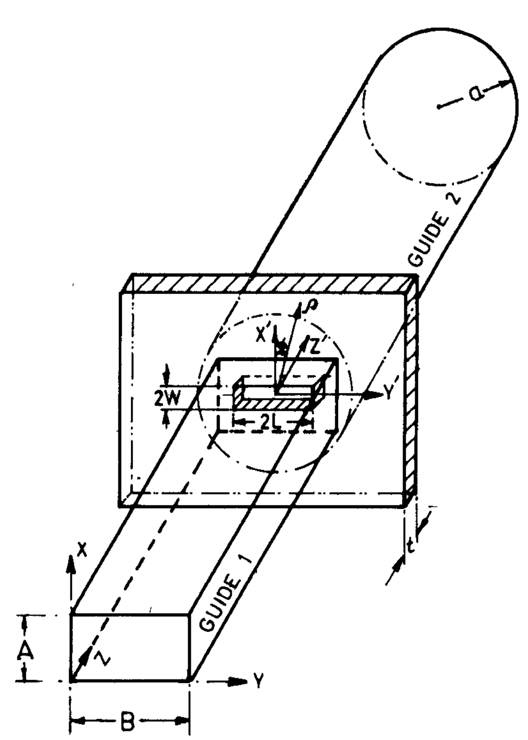

Fig. 1. Slot coupled transition between rectangular and circular waveguides.

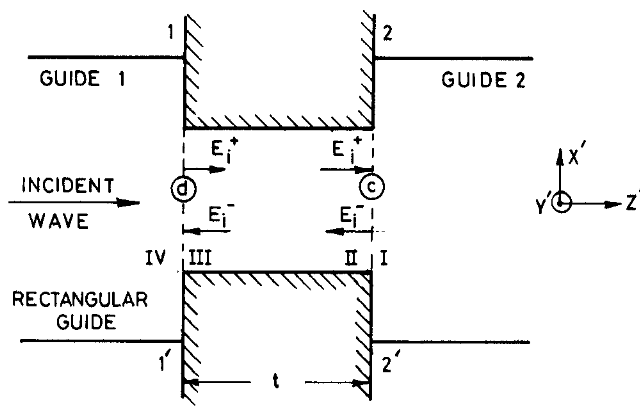

Fig. 2. Expanded view of the coupling slot represented as a slot waveguide.

II. Analysis

Fig. 1 shows the geometry of a slot coupled transition between a rectangular and a circular waveguide. The rectangular coupling slot of length $2L$ and width $2W$ is milled in a metallic plate of thickness t. An expanded view of the slot waveguide representation [7] of the coupling slot is shown in Fig. 2, together with the two interfaces and the incident and reflected waves in the slot waveguide.

Following the procedure of [7, sec. II] the column matrices representing the amplitude coefficients for the total tangential electric fields at the two interfaces of the slot waveguide are same as those given by [7, eqs. (38)–(40)]. They are reproduced